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A Remark with Regard to Inequalities for Some Sums

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Abstract. In this paper we consider some inequalities that are regarded to certain sums.

Inequalities that we shall use in this paper and that are proved in [1] are the following:

(1) $$b^r - a^r \geq r(b-a)(ab)^{\frac{r-1}{2}}, \quad b > a > 0,\ r \geq 1$$

and

(2) $$b^s - a^s \leq s(b-a)(ab)^{\frac{s-1}{2}}, \quad b > a > 0,\ 0 < s \leq 1.$$

If in the inequality (1) we put

(3) $$b = \frac{1}{(n-\frac{1}{2})^2 + x^2 - \frac{1}{4}}, \quad a = \frac{1}{(n+\frac{1}{2})^2 + x^2 - \frac{1}{4}}, \quad x \neq 0,$$

we shall get the inequality

$$\frac{1}{\big((n-\frac{1}{2})^2 + x^2 - \frac{1}{4}\big)^r} - \frac{1}{\big((n+\frac{1}{2})^2 + x^2 - \frac{1}{4}\big)^r} \geq \frac{2nr}{\big((n^2+x^2)^2 - n^2\big)^{\frac{r+1}{2}}} > \frac{2nr}{(n^2+x^2)^{r+1}},$$

where summing for $n = 1, 2, \ldots,$ we get

(4) $$\sum_{n=1}^{\infty} \frac{2n}{(n^2+x^2)^2} < \frac{1}{r(x^2)^r}, \quad r \geq 1,\ x \neq 0.$$

For $r = 1$, from (4) we obtain

(5) $$\sum_{n=1}^{\infty} \frac{2n}{(n^2+x^2)^2} < \frac{1}{x^2}.$$

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If in the inequality (2) we put

$$b = \frac{1}{(n-\frac{1}{2})^2 + x^2 + \frac{1}{4}}, \qquad a = \frac{1}{(n+\frac{1}{2})^2 + x^2 + \frac{1}{4}}, \qquad x \neq 0, \tag{6}$$

we get the inequality

$$\begin{aligned} &\frac{1}{\left((n-\frac{1}{2})^2 + x^2 + \frac{1}{4}\right)^s} - \frac{1}{\left((n+\frac{1}{2})^2 + x^2 + \frac{1}{4}\right)^s} \\ &\leq \frac{2ns}{\left((n^2 + x^2 + \frac{1}{2})^2 - n^2\right)^{\frac{s+1}{2}}} < \frac{2ns}{(n^2+x^2)^{s+1}}, \end{aligned} \tag{7}$$

as it is

$$\frac{1}{(n^2 + x^2 + \frac{1}{2})^2 - n^2} < \frac{1}{(n^2+x^2)^2}.$$

Summing for $n = 1, 2, \ldots$, from (7) we get

$$\frac{1}{s(x^2+\frac{1}{2})^s} < \sum_{n=1}^{\infty} \frac{2n}{(n^2+x^2)^{s+1}}, 0 < s \leq 1, \ x \neq 0. \tag{8}$$

For $s = 1$, from (8) we obtain

$$\frac{1}{x^2+\frac{1}{2}} < \sum_{n=1}^{\infty} \frac{2n}{(n^2+x^2)^2}. \tag{9}$$

Inequalities (5) and (9) written in the form

$$\frac{1}{x^2+\frac{1}{2}} < \sum_{n=1}^{\infty} \frac{2n}{(n^2+x^2)^2} < \frac{1}{x^2}, \quad x \neq 0 \tag{10}$$

represent Mathieu's inequalities (see [2, p. 629]).

For $s = \frac{1}{r}$, $r \geq 1$ $(0 < s \leq 1)$, the inequality (8) reduces to

$$\frac{r}{(x^2+\frac{1}{2})^{\frac{1}{r}}} < \sum_{n=1}^{\infty} \frac{2n}{(n^2+x^2)^{\frac{r+1}{r}}}, \quad r \geq 1. \tag{11}$$

From (4) and (11) we obtain inequality

$$\frac{r}{(x^2+\frac{1}{2})^{\frac{1}{r}}} < \sum_{n=1}^{\infty} \frac{2n}{(n^2+x^2)^{\frac{r+1}{r}}} < \frac{1}{r(x^2)^r}, \quad r \geq 1, \ x \neq 0. \tag{12}$$

If in the inequality (1) we put

$$b = \frac{1}{n-\frac{1}{2}+x^2}, \qquad a = \frac{1}{n+\frac{1}{2}+x^2}, \tag{13}$$

for $r \geq 1$ and x real number, we get inequality

$$\tag{14} \begin{aligned} &\frac{1}{(n-\frac{1}{2}+x^2)^r} - \frac{1}{(n+\frac{1}{2}+x^2)^r} \\ &\geq \frac{r}{\left((n+x^2)^2-\frac{1}{4}\right)^{\frac{r+1}{2}}} > \frac{r}{(n+x^2)^{r+1}}, \end{aligned}$$

where summing for $n = 1, 2, \ldots$, we get inequality

$$\tag{15} \sum_{n=1}^{\infty} \frac{1}{(n+x^2)^{r+1}} < \frac{1}{r(x^2+\frac{1}{2})^{r+1}}, \quad r \geq 1,\ x \text{ is real number.}$$

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